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Tom Cuchta

Nick Wintz

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Research article

Periodic functions related to the Gompertz difference equation

Tom Cuchta[1,*] **and Nick Wintz**[2]

[1] Department of Computer Science and Mathematics, Fairmont State University, Fairmont, WV, USA
[2] Department of Mathematics, Computer Science, and Information Technology, Lindenwood University, St. Charles, MO, USA

* **Correspondence:** Email: tcuchta@fairmontstate.edu; Tel: +1-304-367-4218.

Abstract: We investigate periodicity of functions related to the Gompertz difference equation. In particular, we derive difference equations that must be satisfied to guarantee periodicity of the solution.

Keywords: difference equations; periodic solution; Gompertz; population model

1. Introduction

We study the Gompertz difference equation

$$y^{\Delta}(t) = (\ominus r)(t) y(t) \left(K(t) + a + \int_0^t \frac{y^{\Delta}(\tau)}{y(\tau)} \Delta\tau \right), \quad y(0) = y_0, \tag{1.1}$$

as well as periodic functions that arise from it. This is to say when $\omega \in \{1, 2, \ldots\}$, $f\colon \mathbb{N}_0 \to \mathbb{R}$ is ω-periodic if $f(t + \omega) = f(t)$ for all $t \in \mathbb{N}_0$. Here, $(\ominus r)(t) = \dfrac{-r(t)}{1 + r(t)}$ is the time scales analogue of the growth rate while $K(t)$ is the analogue of the carrying capacity at time t from the traditional continuous Gompertz model. Throughout, we will use notation inspired from time scales calculus for the time scale $\mathbb{T} = \mathbb{N}_0$, including $\sigma(t) = t + 1$, $y^{\Delta}(t) = y(\sigma(t)) - y(t)$, and the integration symbol representing summation, i.e. $\displaystyle\int_a^b f(t)\Delta t = \sum_{k=a}^{b-1} f(k)$. See the monograph [1] for the usual introduction to dynamic equations on time scales and see the recent texts on first and second order boundary value problems on time scales [2] and its companion book on third, fourth, and higher-order boundary value problems on time scales [3] for more recent books.

In [4], the model (1.1) as well as a second model without the $\ominus$ was introduced, solved, and bounds of its solutions were established. Three discrete fractional analogues of (1.1) were explored in [5] by changing the difference to a fractional difference and exploring defining the logarithm with a fractional integral. These three models were compared to another existing fractional Gompertz difference equation [6], which was built around using the classical logarithm instead of a time scales logarithm. The solution of (1.1) can be normalized to create a probability distribution which was studied in [7] where bounds on the expected value were derived and a connection between the classical continuous Gompertz distribution with the q-geometric distribution of the second kind was established. An alternative approach to Gompertz equations on time scales appears in [8] which uses the $\odot$ operation to define a Gompertz dynamic equation.

Gompertz models have been used to study a number of applications in both discrete and continuous settings. This includes studying the growth rate of tumors [9,10], modeling growth of prey in predator-prey dynamics [11], as well as study the change in cost in adopting new technologies [12,13], effect of seasonality for Gompertz models using time series [14], and the spread of COVID-19 [15,16].

2. Preliminaries and definitions

Before introducing our main results, some preliminary definitions and results are in order. Equation (1.1) has the unique solution $y(t) = y_0 e_p(t, 0)$, where

$$p(t) = (\ominus r)(t)\left(a e_{\ominus r}(t,0) - \int_0^t (\ominus r)(s) e_{\ominus r}(t, \sigma(s)) K(s)\Delta s - K(t)\right). \tag{2.1}$$

Here, $e_f\colon \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$, called the discrete exponential, is the unique solution of the initial value problem $y^\Delta = fy, y(0) = 1$. We often make use of the so-called "simple useful formula,"

$$e_p(\sigma(t), 0) = (1 + p(t))e_p(t, 0). \tag{2.2}$$

when rewriting exponentials.

Time scales integration by parts is given by

$$\int_a^b f(\tau)g^\Delta(\tau)\Delta\tau = f(t)g(t)\Big|_a^b - \int_a^b f^\Delta(\tau)g(\sigma(\tau))\Delta\tau. \tag{2.3}$$

A function $f\colon \mathbb{N}_0 \to \mathbb{C}$ is said to be of exponential order α [17, Definition 4.1] if there is an $\alpha \in \mathbb{R}$ with $1 + \alpha > 0$ and a $M > 0$ such that $|f(t)| \le M e_\alpha(t, s)$ for all $t \in \mathbb{N}_0$. In particular, [17, Lemma 4.4] shows that if f is of exponential order α and $|z + 1| > 1 + \alpha$, then $\lim_{t\to\infty} f(t)e_{\ominus z}(t, 0) = 0$.

The time scales Laplace transform is given by [1, Section 3.10]

$$\mathscr{L}\{f\}(z) = \int_0^\infty f(\tau)e_{\ominus z}(\sigma(\tau), 0)\Delta\tau.$$

which for $\mathbb{T} = \mathbb{N}_0$ is a scaled and shifted $\mathcal{Z}$-transform. It's known [18, Theorem 3.2] that if w is a regressive constant and $\mathbb{T} = \mathbb{N}_0$, then

$$\mathscr{L}\{f e_w^\sigma(\cdot, s)\}(z) = \mathscr{L}\{f\}(z \ominus w), \tag{2.4}$$

and if $X(t) = \displaystyle\int_0^t x(\tau)\Delta\tau$, then [18, Theorem 6.4]

$$\mathscr{L}\{X\}(z) = \frac{1}{z}\mathscr{L}\{x\}(z). \tag{2.5}$$

A well-known identity for the $\mathbb{T} = \mathbb{N}_0$ delta derivative operator is

$$f(k+\omega) = \sum_{j=0}^{\omega} \binom{\omega}{j} f^{\Delta^j}(k). \tag{2.6}$$

The Laplace transform for differences of f is given by

$$\mathscr{L}\{f^{\Delta^j}\}(z) = z^j \mathscr{L}\{x\}(z) - \sum_{\ell=0}^{j-1} z^{\ell} f^{\Delta^{j-\ell-1}}(0). \tag{2.7}$$

The Laplace transform of the shifted argument is useful for the sequel.

Lemma 1. *If f is of exponential order α, then*

$$\mathscr{L}\{f(\cdot+\omega)\}(z) = (z+1)^{\omega} \mathscr{L}\{f\}(z) - \sum_{j=0}^{\omega} \binom{\omega}{j} \sum_{\ell=0}^{j-1} z^{\ell} f^{\Delta^{j-\ell-1}}(0). \tag{2.8}$$

Proof. Applying the Laplace transform to (2.6) and using (2.7), we have

$$\begin{aligned}
\mathscr{L}\{f(\cdot+\omega)\}(z) &= \sum_{j=0}^{\omega} \binom{\omega}{j} \mathscr{L}\left\{f^{\Delta^j}\right\}(z) \\
&= \sum_{j=0}^{\omega} \binom{\omega}{j} \left[z^j \mathscr{L}\{f\}(z) - \sum_{\ell=0}^{j-1} z^{\ell} f^{\Delta^{j-\ell-1}}(0) \right] \\
&= \mathscr{L}\{f\}(z) \left(\sum_{j=0}^{\omega} \binom{\omega}{j} z^j \right) - \sum_{j=0}^{\omega} \binom{\omega}{j} \sum_{\ell=0}^{j-1} z^{\ell} f^{\Delta^{j-\ell-1}}(0).
\end{aligned}$$

An application of the binomial theorem to the first summation completes the proof. □

Now we calculate the discrete Laplace transform of a certain time-dependent delta integral.

Lemma 2. *If f is of exponential order α and $X(t) = \int_t^{t+\omega} f(\tau)\Delta\tau$, then for all $z \in \mathbb{C}$ with $|1+z| > 1+\alpha$,*

$$\mathscr{L}\{X\}(z) = \frac{1}{z} \sum_{k=0}^{\omega-1} f(k) + \frac{(z+1)^{\omega} - 1}{z} \mathscr{L}\{f\}(z) - \frac{1}{z} \sum_{j=0}^{\omega} \binom{\omega}{j} \sum_{\ell=0}^{j-1} z^{\ell} f^{\Delta^{j-\ell-1}}(0). \tag{2.9}$$

Proof. First use (2.2) to see

$$e_{\ominus z}(k+1,0) = (1 + (\ominus z))\, e_{\ominus z}(k,0) = \frac{1}{1+z} e_{\ominus z}(k,0).$$

Calculate, where k is understood to be the variable,

$$\left[\sum_{\ell=k}^{k+\omega-1} f(\ell) \right]^{\Delta} = \sum_{\ell=k+1}^{k+\omega} f(\ell) - \sum_{\ell=k}^{k+\omega-1} f(\ell) = f(k+\omega) - f(k).$$

Now since $X(t)=\sum\limits_{\ell=t}^{t+\omega-1} f(\ell)$, (2.2) reveals

$$\mathscr{L}\{X\}(z)=\int_0^\infty\left(\sum_{\ell=\tau}^{\tau+\omega-1} f(\ell)\right)e_{\ominus z}(\sigma(\tau),0)\Delta\tau=\frac{1}{1+z}\int_0^\infty\left(\sum_{\ell=\tau}^{\tau+\omega-1} f(\ell)\right)e_{\ominus z}(\tau,0)\Delta\tau.$$

Since $(\ominus z)$ is constant and $e_{\ominus z}^{\Delta}=(\ominus z)e_{\ominus z}$, we observe

$$\mathscr{L}\{X\}(z)=\frac{1}{1+z}\frac{1}{(\ominus z)}\int_0^\infty\left(\sum_{\ell=\tau}^{\tau+\omega-1} f(\ell)\right)e_{\ominus z}^{\Delta}(\tau,0)\Delta\tau=-\frac{1}{z}\int_0^\infty\left(\sum_{\ell=\tau}^{\tau+\omega-1} f(\ell)\right)e_{\ominus z}^{\Delta}(\tau,0)\Delta\tau.$$

Apply (2.3) to obtain

$$\mathscr{L}\{X\}(z)=-\frac{1}{z}\left(\sum_{\ell=\tau}^{\tau+\omega-1} f(\ell)\right)e_{\ominus z}(\tau,0)\Bigg|_{\tau=0}^{\tau=\infty}+\frac{1}{z}\int_0^\infty\Big(f(\tau+\omega)-f(\tau)\Big)e_{\ominus z}(\sigma(\tau),0)\Delta\tau.$$

Thus we have

$$\mathscr{L}\{X\}(z)=\frac{1}{z}\sum_{k=0}^{\omega-1} f(k)+\frac{1}{z}\mathscr{L}\{f(\cdot+\omega)\}(z)-\frac{1}{z}\mathscr{L}\{f\}(z).$$

Applying (2.8) to the middle term of the right-hand side completes the proof. □

3. Periodicity of p

First we establish which functions r yield $e_{\ominus r}$ to be ω-periodic.

Lemma 3. *The discrete exponential is periodic, meaning*

$$e_{\ominus r}(t+\omega,0)=e_{\ominus r}(t,0) \tag{3.1}$$

if and only if

$$r(t+\omega-1)=-1+\prod_{k=t}^{t+\omega-2}\frac{1}{1+r(k)}. \tag{3.2}$$

Proof. First calculate

$$1+(\ominus r)(t)=1-\frac{r(t)}{1+r(t)}=\frac{1}{1+r(t)}.$$

Now, (3.1) becomes

$$\prod_{k=0}^{t+\omega-1}\frac{1}{1+r(k)}=\prod_{k=0}^{t-1}\frac{1}{1+r(k)},$$

hence $1=\prod\limits_{k=t}^{t+\omega-1}\frac{1}{1+r(k)}$, which is equivalent to $r(t+\omega-1)=-1+\prod\limits_{k=t}^{t+\omega-2}\frac{1}{1+r(k)}$. Since all steps are reversible, the proof is complete. □

Since the ω-periodicity of $e_{\ominus r}$ is equivalent to r satisfying the difference equation (3.2), solving it is of importance.

Lemma 4. *If* $r(0),\dots,r(\omega-2)$ *are known, then the unique solution of* (3.2) *is* ω*-periodic.*

Proof. Use (3.2) with $t=0$ to generate the ωth value

$$r(\omega-1)=-1+\frac{1}{(1+r(0))(1+r(1))\dots(1+r(\omega-2))}.$$

We claim that the function r is ω-periodic. From (3.2), we obtain

$$r(t+\omega)\quad=-1+\prod_{k=t+1}^{t+\omega-1}\frac{1}{1+r(k)}=-1+\frac{1}{(1+r(t+1))(1+r(t+2))\dots(1+r(t+\omega-1))}.$$

But also by (3.2),

$$1+r(t+\omega-1)=\prod_{k=t}^{t+\omega-2}\frac{1}{1+r(k)}=\frac{1}{(1+r(t))(1+r(t+1))\dots(1+r(t+\omega-2))}.$$

Therefore,

$$\begin{aligned}r(t+\omega)&=-1+\frac{1}{(1+r(t+1))(1+r(t+2))\dots(1+r(t+\omega-2))\left[\frac{1}{(1+r(t))\dots(1+r(t+\omega-2))}\right]}\\&=-1+\frac{1}{\frac{1}{1+r(t)}}=r(t),\end{aligned}$$

completing the proof. ◻

By (2.1), when p is ω-periodic, $p(t+\omega)=p(t)$ expands to

$$\begin{aligned}(\ominus r)(t+\omega)&\left(ae_{\ominus r}(t+\omega,0)-\int_0^{t+\omega}(\ominus r)(s)e_{\ominus r}(t+\omega,\sigma(s))K(s)\Delta s-K(t+\omega)\right)\\&=(\ominus r)(t)\left(ae_{\ominus r}(t,0)-\int_0^{t}(\ominus r)(s)e_{\ominus r}(t,\sigma(s))K(s)\Delta s-K(t)\right).\end{aligned}\tag{3.3}$$

Theorem 5. *If* r *solves* (3.2)*, then* $p(t+\omega)=p(t)$ *if and only if*

$$K(t+\omega)=K(t)-\int_t^{t+\omega}(\ominus r)(s)e_{\ominus r}(t+\omega,\sigma(s))K(s)\Delta s.$$

Proof. By Lemma 3, $e_{\ominus r}(\cdot,0)$ is ω-periodic. By Lemma 4, r is ω-periodic, hence $(\ominus r)$ is also ω-periodic. Using (3.3), we divide by $(\ominus r)(t+\omega)$ and subtract $ae_{\ominus r}(t+\omega,0)$ to obtain

$$-\int_0^{t+\omega}(\ominus r)(s)e_{\ominus r}(t+\omega,\sigma(s))K(s)\Delta s-K(t+\omega)=-\int_0^{t}(\ominus r)(s)e_{\ominus r}(t,\sigma(s))K(s)\Delta s-K(t).$$

Hence

$$0 = K(t+\omega) - K(t) + \int_0^t (\ominus r)(s)\Big[e_{\ominus r}(t+\omega,\sigma(s)) - e_{\ominus r}(t,\sigma(s))\Big]K(s)\Delta s + \int_t^{t+\omega} (\ominus r)(s) e_{\ominus r}(t+\omega,\sigma(s))\Delta s. \quad (3.4)$$

By the semigroup property and the periodicity of $e_{\ominus r}(\cdot, 0)$,

$$e_{\ominus r}(t+\omega,\sigma(s)) - e_{\ominus r}(t,\sigma(s)) = \Big[e_{\ominus r}(t+\omega,0) - e_{\ominus r}(t,0)\Big]e_{\ominus r}(0,\sigma(s)) = 0, \quad (3.5)$$

and applying (3.5) to (3.4) completes the proof. □

Theorem 6. *If $r(t) = r$ is constant and K is ω-periodic, then $p(t+\omega) = p(t)$ if and only if*

$$K(t+\omega-1) = \frac{a}{r(1+r)^t}\left[1 - \frac{1}{(1+r)^\omega}\right] + \frac{1}{1+r}\int_0^t \frac{K(s)}{(1+r)^{t-s-1}}\Delta s - \frac{1}{1+r}\int_0^{t+\omega-1} \frac{K(s)}{(1+r)^{t+\omega-s-1}}\Delta s.$$

Proof. From (3.3), since r is constant, so is $(\ominus r)$, hence both $(\ominus r)(t+\omega)$ and $(\ominus r)(t)$ can be divided off. Similarly, since $K(t+\omega) = K(t)$, those terms also vanish in (3.3). What remains is

$$ae_{\ominus r}(t+\omega,0) - (\ominus r)\int_0^{t+\omega} e_{\ominus r}(t+\omega,\sigma(s))K(s)\Delta s = ae_{\ominus r}(t,0) - (\ominus r)\int_0^t e_{\ominus r}(t,\sigma(s))K(s)\Delta s$$

Thus,

$$\frac{a}{(1+r)^{t+\omega}} + \frac{r}{1+r}\int_0^{t+\omega} \frac{K(s)}{(1+r)^{t+\omega-s-1}}\Delta s = \frac{a}{(1+r)^t} + \frac{r}{1+r}\int_0^t \frac{K(s)}{(1+r)^{t-\sigma(s)}}\Delta s$$

Now

$$\frac{a}{(1+r)^{t+\omega}} + \frac{r}{1+r}\int_0^{t+\omega-1} \frac{K(s)}{(1+r)^{t+\omega-s-1}}\Delta s + rK(t+\omega-1) = \frac{a}{(1+r)^t} + \frac{r}{1+r}\int_0^t \frac{K(s)}{(1+r)^{t-s-1}}\Delta s,$$

and solving for $K(t+\omega-1)$ completes the proof. □

4. Periodicity of e_p

Define $\alpha(t,s) := (\ominus r)(s)e_{\ominus r}(t,\sigma(s))K(s)$ and

$$\beta(t) := \frac{1}{(\ominus r)(t+\omega-1)}\left[1 - \frac{1}{e_p(t+\omega-1,t)}\right] + ae_{\ominus r}(t+\omega-1,0).$$

Theorem 7. *If $r\colon \mathbb{N}_0 \to \mathbb{R}$, then the function $t \mapsto e_p(t,0)$ is ω-periodic if and only if*

$$K(t+\omega-1) = \beta(t) - \int_0^{t+\omega-1} \alpha(t+\omega-1,s)\Delta s. \quad (4.1)$$

Proof. If e_p is ω-periodic, then using the semigroup property of e_p, we obtain

$$p(t+\omega-1) = -1 + \frac{1}{e_p(t+\omega-1,t)}.$$

By (2.1), this becomes

$$(\ominus r)(t+\omega-1)\left[ae_{\ominus r}(t+\omega-1,0) - \int_0^{t+\omega-1}\alpha(t+\omega-1,s)\Delta s \right.$$
$$\left. - K(t+\omega-1)\right] = -1 + \frac{1}{e_p(t+\omega-1,t)}, \quad (4.2)$$

which we rearrange to obtain (4.1). All steps are reversible so the converse is also true, completing the proof. □

We provide a numerical example of Theorem 7 in Figure 1. It is difficult in general to solve (4.1) in closed form, but if r is a constant function, then it may be solved with Laplace transform techniques.

Theorem 8. *If $r \in \mathcal{R}_c(\mathbb{N}_0, \mathbb{R})$ and K is of exponential order α, then for all $|z+1| > 1+\alpha$, the Laplace transform of* (4.1) *is*

$$\mathscr{L}\{K\}(z) = \frac{1}{(z+1)^{\omega-1} - \dfrac{r\Big((z\oplus r)+1\Big)^{\omega-1}}{(1+r)^{\omega-1}(z\oplus r)}} \times \left[\mathscr{L}\{\beta\}(z) + \frac{r}{(1+r)^{\omega-1}(z\oplus r)}\sum_{k=0}^{\omega-2} e_r(\sigma(k),0)K(k)\right.$$
$$\left. + \sum_{j=0}^{\omega-1}\binom{\omega-1}{j}\sum_{\ell=0}^{j-1} z^{\ell}\left[K^{\Delta^{k-\ell-1}}(0) - \frac{r}{(1+r)^{\omega-1}(z\oplus r)}\left[e_r(\sigma(\cdot),0)K(\cdot)\right]^{\Delta^{k-\ell-1}}(0)\right]\right].$$

Proof. By the semigroup and reciprocal properties for the discrete exponential, (4.1) becomes

$$K(t+\omega-1) = \beta(t) - (\ominus r)e_{\ominus r}(t+\omega-1,0)\int_0^{t+\omega-1} e_r(\sigma(s),0)K(s)\Delta s.$$

By (2.8), we know that

$$\mathscr{L}\{K(\cdot+\omega-1)\}(z) = (z+1)^{\omega-1}\mathscr{L}\{K\}(z) - \sum_{j=0}^{\omega-1}\binom{\omega-1}{j}\sum_{\ell=0}^{j-1} z^{\ell}K^{\Delta^{j-\ell-1}}(0)$$

Using (2.2), compute

$$e_{\ominus r}(t+\omega-1,0) = e_{\ominus r}(\sigma^{\omega-1}(t),0) = (1+(\ominus r))^{\omega-2}\, e_{\ominus r}(t+1,0) = \frac{e_{\ominus r}(t+1,0)}{(1+r)^{\omega-2}}. \quad (4.3)$$

Let

$$\begin{aligned} g(t) &= (\ominus r)e_{\ominus r}(t+\omega-1,0)\int_0^t e_r(\sigma(s),0)K(s)\Delta s \\ &= \frac{-r}{1+r}e_{\ominus r}(t+\omega-1,0)\int_0^t e_r(\sigma(s),0)K(s)\Delta s \\ &= \frac{-r}{(1+r)^{\omega-1}}e_{\ominus r}(t+1,0)\int_0^t e_r(\sigma(s),0)K(s)\Delta s \end{aligned}$$

Using (2.4), (2.5), and (4.3), we compute

$$\begin{aligned}\mathscr{L}\{g\}(z) &= \frac{-r}{(1+r)^{\omega-1}}\mathscr{L}\left\{\int_0^{\cdot} e_r(\sigma(s),0)K(s)\Delta s\right\}(z\oplus r)\\ &= \frac{-r}{(1+r)^{\omega-1}(z\oplus r)}\mathscr{L}\left\{e_r(\sigma(\cdot),0)K(\cdot)\right\}(z\oplus r)\\ &= \frac{-r}{(1+r)^{\omega-1}(z\oplus r)}\mathscr{L}\left\{K\right\}(z).\end{aligned}$$

Now let $h(t) = \displaystyle\int_t^{t+\omega-1} (\ominus r)(s)e_{\ominus r}(t+\omega-1,\sigma(s))K(s)\Delta s$. Using (2.9),

$$\begin{aligned}\mathscr{L}\{h\}(z) &= \frac{-r}{1+r}\mathscr{L}\left\{\int_{\cdot}^{\cdot+\omega-1} e_{\ominus r}(\cdot+\omega-1,\sigma(s))K(s)\Delta s\right\}(z)\\ &= \frac{-r}{(1+r)^{\omega-1}}\mathscr{L}\left\{e_{\ominus r}(\sigma(\cdot),0)\int_{\cdot}^{\cdot+\omega-1} e_r(\sigma(s),0)K(s)\Delta s\right\}(z)\\ &= \frac{-r}{(1+r)^{\omega-1}}\mathscr{L}\left\{\int_{\cdot}^{\cdot+\omega-1} e_r(\sigma(s),0)K(s)\Delta s\right\}(z\oplus r)\\ &= \frac{-r}{(1+r)^{\omega-1}}\left[\frac{1}{z\oplus r}\sum_{k=0}^{\omega-2} e_r(\sigma(k),0)K(k) + \frac{\left((z\oplus r)+1\right)^{\omega-1}-1}{z\oplus r}\mathscr{L}\left\{e_r(\sigma(\cdot),0)K(\cdot)\right\}(z\oplus r)\right.\\ &\quad \left.-\frac{1}{z\oplus r}\sum_{j=0}^{\omega-1}\binom{\omega-1}{j}\sum_{\ell=0}^{j-1} z^{\ell}\Big[e_r(\sigma(\cdot),0)K(\cdot)\Big]^{\Delta^{j-\ell-1}}(0)\right]\\ &= \frac{-r}{(1+r)^{\omega-1}(z\oplus r)}\left[\sum_{k=0}^{\omega-2} e_r(k+1,0)K(k) + \left((z\oplus r)+1\right)^{\omega-1}-1\Big)\mathscr{L}\left\{e_r(\sigma(\cdot),0)K(\cdot)\right\}(z\oplus r)\right.\\ &\quad \left.-\sum_{j=0}^{\omega-1}\binom{\omega-1}{j}\sum_{\ell=0}^{j-1} z^{\ell}\Big[e_r(\sigma(\cdot),0)K(\cdot)\Big]^{\Delta^{j-\ell-1}}(0)\right].\end{aligned}$$

One further step applying (2.8) on the second term yields

$$\begin{aligned}\mathscr{L}\{h\}(z) &= \frac{-r}{(1+r)^{\omega-1}(z\oplus r)}\left[\sum_{k=0}^{\omega-2} e_r(\sigma(k),0)K(k) + \left(\left((z\oplus r)+1\right)^{\omega-1}-1\right)\mathscr{L}\left\{K\right\}(z)\right.\\ &\quad \left.-\sum_{j=0}^{\omega-1}\binom{\omega-1}{j}\sum_{\ell=0}^{j-1} z^{\ell}\Big[e_r(\sigma(\cdot),0)K(\cdot)\Big]^{\Delta^{j-\ell-1}}(0)\right].\end{aligned}$$

Therefore we have shown that the Laplace transform of (4.1) is

$$\begin{aligned}(z+1)^{\omega-1}\mathscr{L}\{K\}(z) - \sum_{j=0}^{\omega-1}\binom{\omega-1}{j}\sum_{\ell=0}^{j-1} z^{\ell}\Delta^{j-\ell-1}K(0) = \mathscr{L}\{\beta\}(z)\\ + \frac{r}{(1+r)^{\omega-1}(z\oplus r)}\left[\sum_{k=0}^{\omega-2} e_r(k+1,0)K(k) + \left((z\oplus r)+1\right)^{\omega-1}\mathscr{L}\left\{K\right\}(z)\right.\end{aligned}$$

$$
-\sum_{j=0}^{\omega-1}\binom{\omega-1}{j}\sum_{\ell=0}^{j-1}z^{\ell}\Big[e_r(\sigma(\cdot),0)K(\cdot)\Big]^{\Delta^{j-\ell-1}}(0)\Bigg].
$$

Solving for $\mathscr{L}\{K\}(z)$ completes the proof. □

Now we consider the reverse case of Theorem 7 where K is given and r must be solved for.

Theorem 9. *If $K\colon \mathbb{N}_0 \to \mathbb{R}$ is known, then the function $t \mapsto e_p(t,0)$ is ω-periodic if and only if*

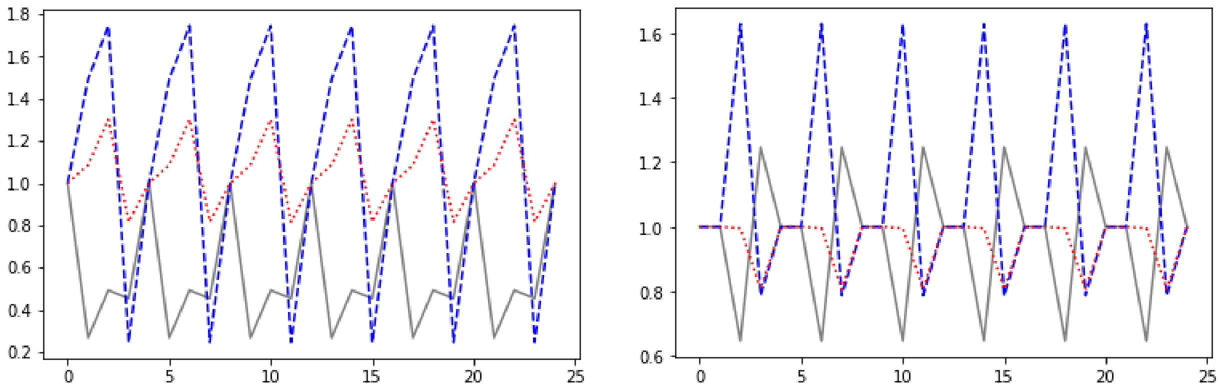
$$
r(t+\omega-1) = \ominus\left(\frac{-1+\dfrac{1}{e_p(t+\omega-1,t)}}{ae_{\ominus r}(t+\omega-1,0)-\displaystyle\int_0^{t+\omega-1}\alpha(t+\omega-1,s)\Delta s - K(t+\omega-1)}\right).
$$

Proof. By solving (4.2) for $(\ominus r)(t+\omega-1)$, we obtain

$$
(\ominus r)(t+\omega-1) = \frac{-1+\dfrac{1}{e_p(t+\omega-1,t)}}{ae_{\ominus r}(t+\omega-1,0)-\displaystyle\int_0^{t+\omega-1}\alpha(t+\omega-1,s)\Delta s - K(t+\omega-1)},
$$

and so taking $\ominus$ of both sides completes the proof, since all steps are algebraically reversible. □

We provide a numerical example of Theorem 9 in Figure 2.

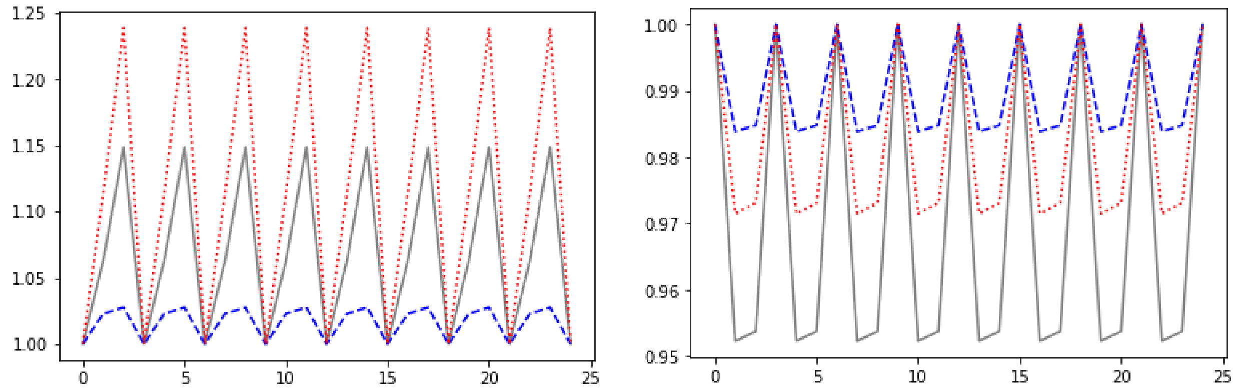


(a) $r(t) = 3$ **(b)** $r(t) = t(2+\sin(t))$

Figure 1. As an application of Theorem 7, three 4-periodic solutions of (1.1) with initial condition $y(0) = 1$ are plotted for given r and randomly selected initial values for $K(0)$, $K(1)$, and $K(2)$ chosen from the interval $(0,2)$.

5. Conclusion

We have explored periodicity of functions related to the Gompertz difference equation (1.1). In Theorem 5, we found a difference equation that K must satisfy in order for p to be ω-periodic whenever



(a) $K(t) = 3$ **(b)** $K(t) = t$

Figure 2. As an application of Theorem 9, three 4-periodic solutions of (1.1) with initial condition $y(0) = 1$ for given K and randomly selected initial values for $r(0)$, $r(1)$, and $r(2)$ chosen from the interval $(0, 0.1)$.

r is itself ω-periodic. Theorem 6 does the same thing, but when r is constant. In Theorem 7, we considered ω-periodicity of solutions of (1.1) and arrived at difference equations that K must solve in order to guarantee it. In Theorem 8 we solved that difference equation in the special case of a constant r using Laplace transform techniques. Finally, in Theorem 9, we instead found a difference equation that r must solve if K is known.

Future work in this area includes the extension of the results to ω-periodic functions on more general time scales as studied in [19,20]. Throughout, we have showcased the basic framework for these results on a more general time scale to aid in such a generalization. The connections between Volterra integral equations and generalizations of (1.1) are of interest, as well as interpreting the function K as a periodic control for population models.

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Conflict of interest

The authors declare that there is no conflict of interest.

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