Optimal control for delayed nanoparticle dosing models using conformable derivatives

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Abstract

Previously, a research group used a theoretical state-space framework to study the effects nanoparticles had on cancerous tumors in mice. Since sensors are used to determine the number of nanoparticles found in the bloodstream, it is natural for a time delay to occur when making adjustments to the dosage. Here, an equal time delay has been put on the state and control to study the effects of multiple dosage strategies. However, these researchers did not consider how the absorption of these nanoparticles would affect treatment. In this project, we construct a model using a conformable derivative first introduced by Khalil et al in 2014. This time-weighted derivative has many of the same properties as the classical derivative but lacks the semigroup property for the exponential. Here, we find an optimal control that minimizes a given cost. This control is propagated by a pseudo-Riccati equation, which itself includes a time delay.

Definition (Khalil Derivative)

Let $f:[0,\infty)\to\mathbb{R}$ and let $\alpha\in(0,1]$. Then the conformable derivative of order α of f at t is defined by

$$f^{(\alpha)}(t) := \begin{cases} \lim\limits_{\theta\to 0} \dfrac{f(t+\theta t^{1-\alpha})-f(t)}{\theta}, & t>0 \\ \lim\limits_{s\to 0^+} f^{(\alpha)}(s), & t=0, \end{cases}$$

provided that the limit exists.

Properties of the Conformable Derivative

Let $\alpha\in(0,1]$. Let f,g be $\alpha-$differentiable for $t>0$, and let $a,b\in\mathbb{R}$. Then

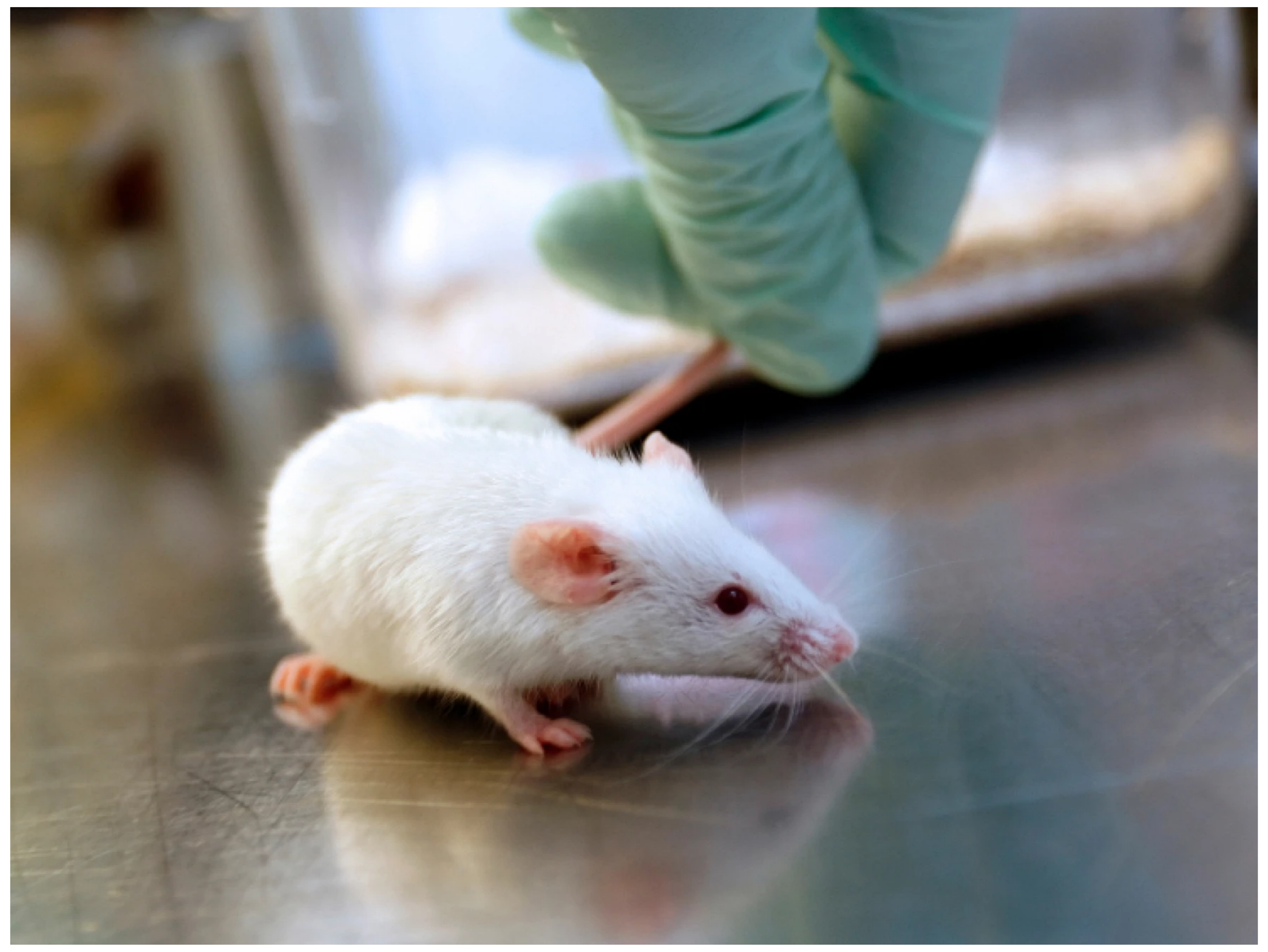
- $(af+bg)^{(\alpha)}(t) = af^{(\alpha)}(t)+bg^{(\alpha)}(t)$,
- $(t^b)^{(\alpha)} = bt^{b-\alpha}$,
- $(b)^{(\alpha)} = 0$,
- $(fg)^{(\alpha)}(t) = f^{(\alpha)}(t)g(t)+f(t)g^{(\alpha)}(t)$,
- $\left(\dfrac{f}{g}\right)^{(\alpha)}(t) = \dfrac{g(t)f^{(\alpha)}(t)-f(t)g^{(\alpha)}(t)}{[g(t)]^2}$, and
- if f is differentiable, then $f^{(\alpha)}(t) = t^{1-\alpha}f'(t)$.

Definition (Khalil Integral)

Let $\alpha\in(0,1]$. The α-conformable integral of f is defined by

$$I_\alpha^a(f)(t) := \int_a^t \frac{f(\tau)}{\tau^{1-\alpha}}\mathrm{d}\tau,$$

where the integral here is the usual Riemann integral.



Delayed Model

Suppose nanoparticles are injected into various tissues of theoretical lab mice. For practical considerations, there is a time delay that corresponds to the injection of the nanoparticles and the initial conditions used for additional injections/readings. As a result, we have

$$x^{(\alpha)}(t) = Ax(t-h)+Bu(t-h)$$

where

- $x\in\mathbb{R}^n$ represents the state, the concentration of nanoparticles in the bloodstream
- $u\in\mathbb{R}^m$ represents the control
- h is the corresponding time delay

We make the natural assumption that our system is completely observable.

Delayed Conformable Linear Quadratic Regulator (DCLQR)

System: $x^{(\alpha)}(t) = Ax(t-h)+Bu(t-h), \quad x(0)=x_0$

Cost: $J(x,u) = \frac{1}{2}x^T(t_f)S(t_f)x(t_f)+\frac{1}{2}\int_0^{t_f}\frac{[x^TQx+u^TRu](\tau)}{\tau^{1-\alpha}}\mathrm{d}\tau,$

where $S(t_f)$,$Q\geq 0$, $R>0$

Feedback Gain: $K(t) := R^{-1}B^TS(t)$

Quasi-Riccati Equation:

$$-S^{(\alpha)}(t) = A^TN^TS(t)+S(t)AN-S(t)BMR^{-1}M^TNS(t-h)+Q,$$

where $M=\dfrac{\partial u_1}{\partial u}$ and $N=\dfrac{\partial x_1}{\partial x}$

Optimal Control: $u^*(t) = -K(t)x(t)$

Optimal Cost: $J^*(x,u) = \frac{1}{2}x^T(0)S(0)x(0)$

Current Status

This framework allows us to formulate of a quasi-Riccati equation necessary to determine an optimal control, something other generalized/ fractional derivatives cannot do as a result of their limitations. We also demonstrate a delay in this equation not previously expressed in the original publication by Basin et al. At present, there are difficulties in providing simulations for the theoretical results as the state also includes a delay instead of just the control. This requires more finesse.

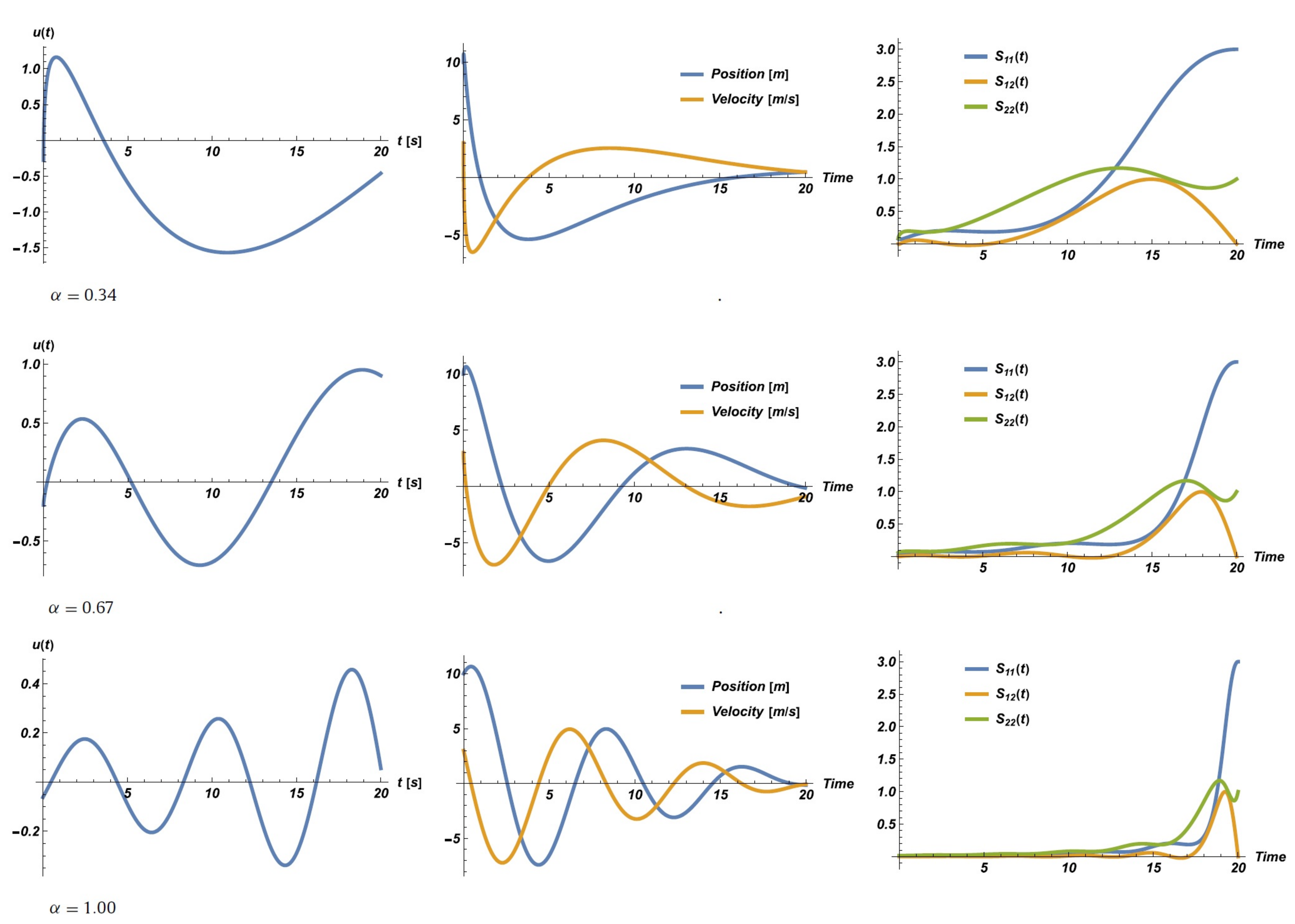
Damped Oscillator with No Delay

Consider the control system

$$x^{(\alpha)}(t) = \begin{bmatrix}0 & 1\\ -0.64 & -0.16\end{bmatrix}x(t)+\begin{bmatrix}0\\1\end{bmatrix}u(t), \quad x_0=\begin{bmatrix}10\\3\end{bmatrix}$$

associated with the quadratic cost functional

$$J(x,u) = \frac{1}{2}x^T(t_f)S(t_f)x(t_f)+\frac{1}{2}\int_0^{50}\frac{\left(x^T\begin{bmatrix}0.001 & 0\\ 0 & 0.001\end{bmatrix}x+u^2\right)(\tau)}{\tau^{1-\alpha}}\mathrm{d}\tau.$$



Related/Future Projects

- Conformable Kalman filter (REU group, Summer 2023)
- Confromable information filter (S. Hungerford and J. Smith, Space Grant Project 2024)
- Conformable Heart Rate Controller (with T. Cuchta and Ö. Öztürk)
- Linear quadratic pursuit-evasion games (time scales version with D. Funk and R. Williams)
- Steady-state results
- Communications models
- Conformable LQR with multiple delay in state/control
- Tracking of economic indicators

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