Over-determined control systems on time scales

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Abstract

Control theory is theory is a branch of mathematics focused on observing or controlling a process governed by a dynamic equation. We use state-space notation to represent all meaningful information about of our process. This means our processes are expressed in vector form. Typically, the state matrix is square. In this project, we consider a control system where the corresponding state is over-determined, meaning there are more rows than columns. In addition, our state equation is on a time scale $\mathbb{T}$, which allows us to consider discrete, continuous, or hybrid measurements. Here, we offer two methods to solve the dynamic system. Finally, we offer numerical results to a corresponding electrical power system.

Definition (Time Scales)

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. Examples of time scales include $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = h\mathbb{Z}$ for $h > 0$, the quantum numbers $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^k : k \in \mathbb{Z}\} \cup \{0\}$ for $q > 1$, $\mathbb{T} = \mathbb{P}_{a,b} = \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a]$, for $a,b > 0$, and the Cantor set.

Time Scale	Derivative	Integral
$\mathbb{R}$	$f'(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s}$	$\int_a^b f(t)\,\mathrm{d}t$
$h\mathbb{Z}$	$\Delta_h f(t) := \frac{f(t+h)-f(t)}{h}$	$\sum_{t=a/h}^{b/h-1} f(ht)h$
$\mathbb{T}_{iso}$	$f^{\Delta}(t) = \frac{f(\sigma(t))-f(t)}{\mu(t)}$	$\sum_{t\in[a,b)} f(t)\mu(t)$

where

- Forward shift: $\sigma(t)$ is the next available point in $\mathbb{T}$.
- Graininess: $\mu(t) = \sigma(t) - t$.

The Model

Consider the dynamic over-determined state equation

$$Ex^{\Delta}(t) = Ax(t) + u(t), x(0) = x_0,$$

where

- $x \in \mathbb{R}^n$ represents the state
- $u \in \mathbb{R}^m$ represents the control
- E and A are $m \times n$ matrices such that $m > n$.

Here, we are not necessarily guaranteed a unique solution.

Method 1: Frequency Domain Approach

Using the time scale analog of the Laplace transform results in the system

$$(zE - A)X(z) = Ex_0 + U(z),$$

where the matrix polynomial $sE - A$ is called the pencil of the model. We introduce a matrix function $P(z)$ such that

$$P(z)(zE - A) = \begin{bmatrix} \hat{A}(z) \\ \mathcal{O} \end{bmatrix}$$

where $\hat{A}(z)$ is a square matrix. The goal is to express a solution of the model in terms of $\hat{A}$.

Method 2: Time Domain Approach

For matrices E where it exists, we apply the pseudo-inverse

$$E^{+} = (E^T E)^{-1} E^T$$

to both sides to obtain the corresponding square control system

$$x^{\Delta}(t) = Fx(t) + v(t).$$

We then seek numerical results.

Classical Electromechanical Model

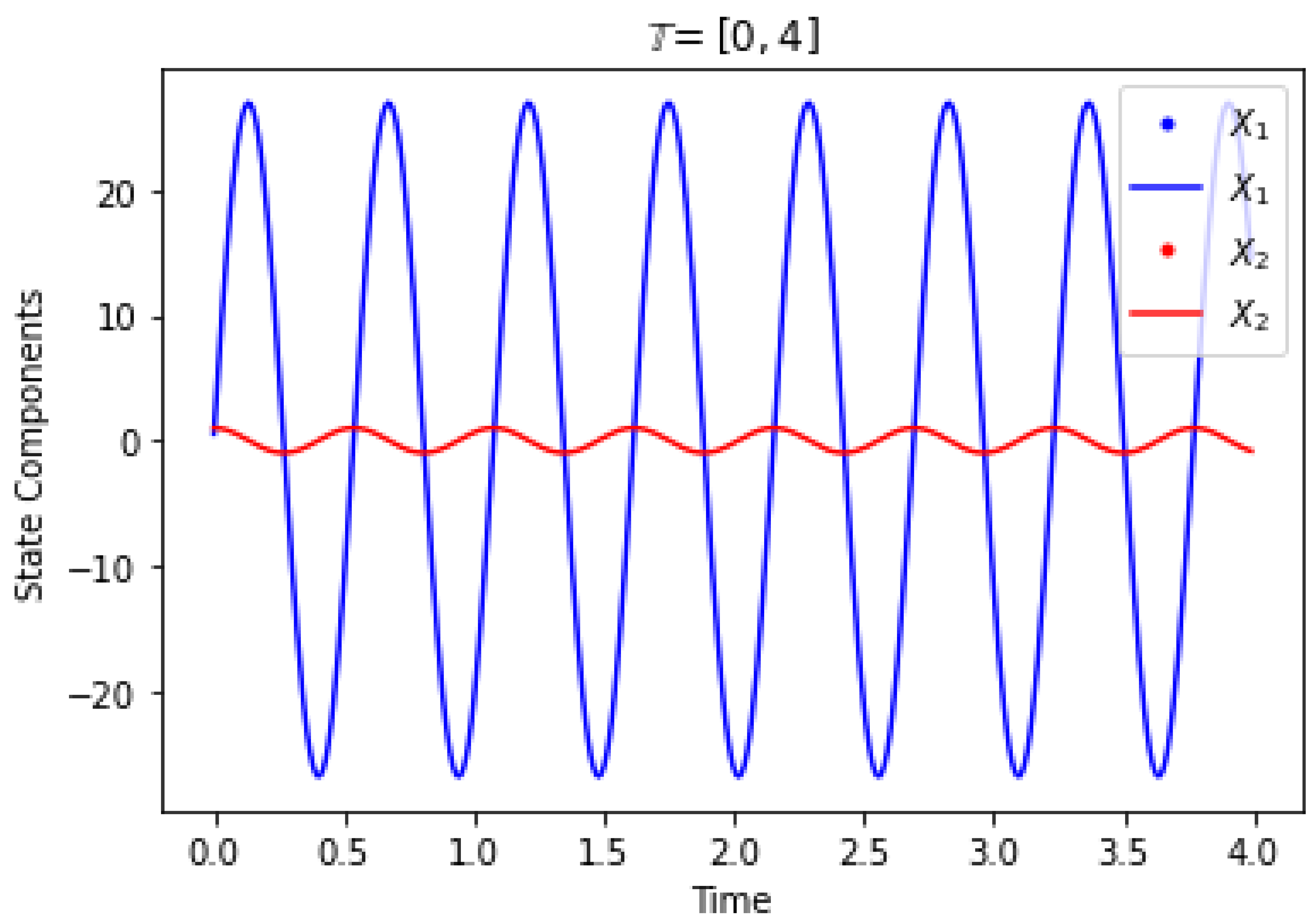
Consider the classical synchronous machine model

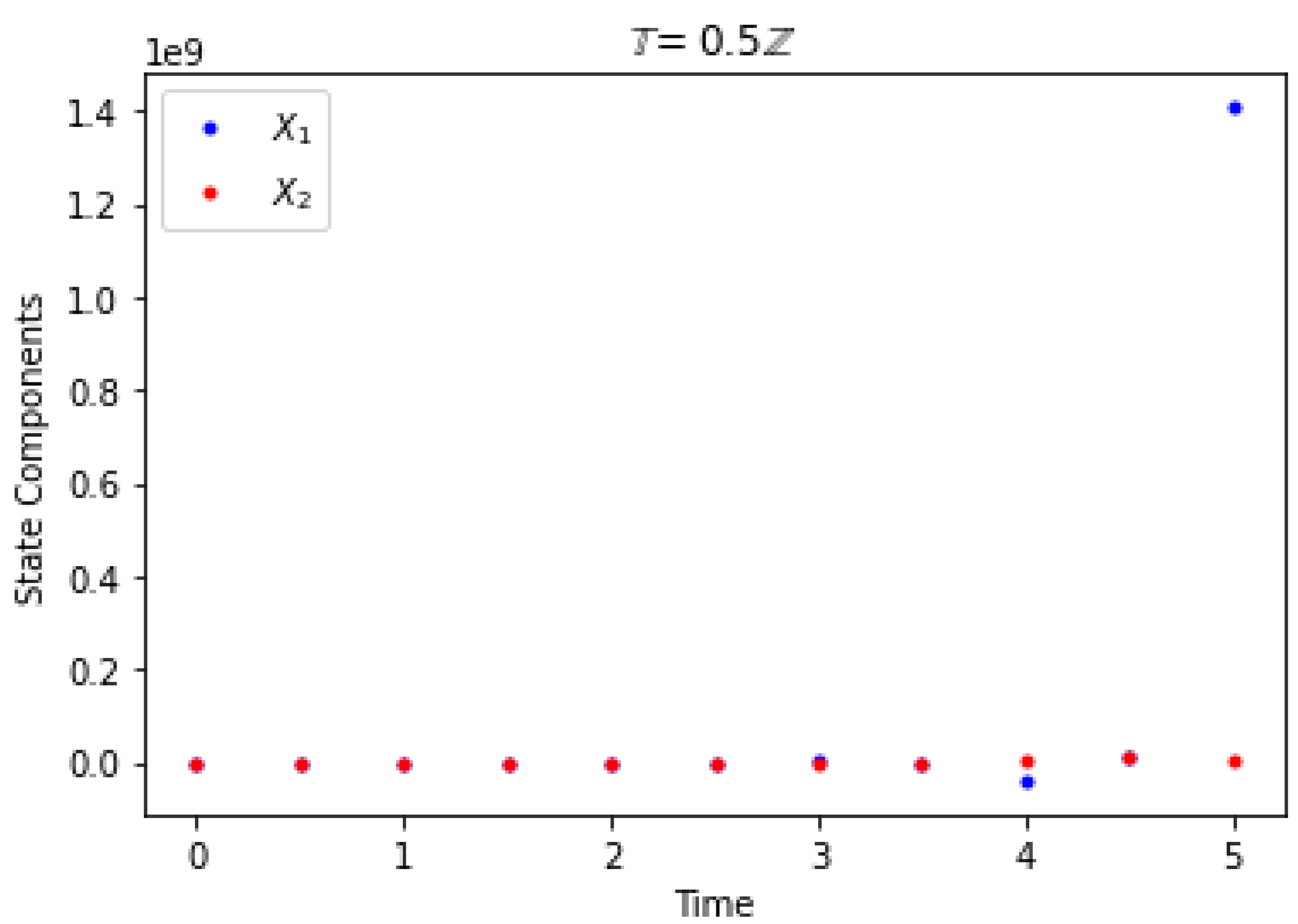
$$\begin{bmatrix} \frac{1}{100\pi} & 0 \\ 0 & 14 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{\Delta}(t) = \begin{bmatrix} 0 & 1 \\ -\sqrt{3} & 0 \\ 0 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta P_m \\ -\delta \cdot \Delta P_m \end{bmatrix}, \quad x(0) = \begin{bmatrix} \frac{\pi}{6} \\ 1 \end{bmatrix}$$

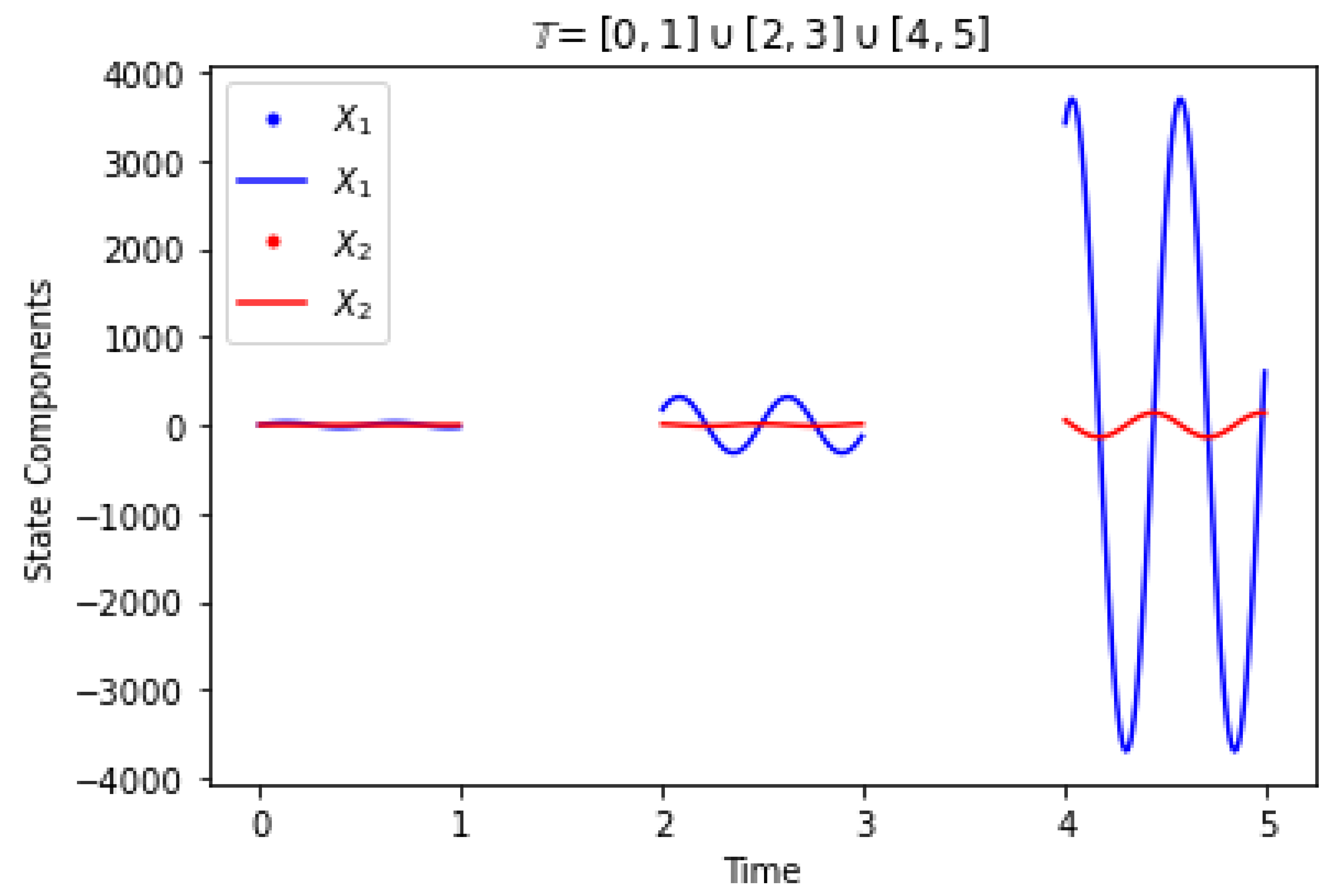
where

- x_1 is the rotor's angular position
- x_2 is the rotor's angular speed
- $P_m = 1$ pu(MW) is the mechanical power
- ΔP_m is the change in mechanical power
- $\delta = 0$ is the equilibrium angular position

Note: in this setting, the damping has been removed.







No Solution

Note that the control system

$$Ex^{\Delta}(t) = Ax(t)$$

where

$$E = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 1 & 2 & 2 \\ 0 & 2 & 3 \end{bmatrix}$$

has no solution since

$$E^T E = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}$$

does not have an inverse.

Related/Future Projects

- Formally introduce the necessary conditions for a unique solution.
- Be able to generalize the usual regressivity condition for the state matrix A based on corresponding Jordan blocks.
- Be able to express a matrix exponential in terms of these Jordan blocks.
- Find the controllability and observability conditions for these models on time scales.
- Establish the optimal control conditions/properties for these models on time scales.
- Establish the Kalman filter for these models on time scales.

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