Conformable Regulator Problems with Fixed Delay

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Abstract

In this project, we consider processes guided by a conformable derivative first introduced by Khalil et al in 2014. This time-weighted derivative has many of the same properties as the classical derivative but lacks the semigroup property for the exponential. Here, we study a conformable linear system where the state and control are subject to the same fixed delay. Our process is also subject to wear and tear, represented by a cost functional. Our goal is to find an optimal control that minimizes this cost. This control is propagated by a quasi-Ricatti equation, which itself includes a time delay. Finally, we offer a physical model associated with our delayed as well as numerical simulations for different rates alpha.

Definition (Khalil Derivative)

Let $f : [0, \infty) \to \mathbb{R}$ and let $\alpha \in (0, 1]$. Then the conformable derivative of order α of f at t is defined by

$$f^{(\alpha)}(t) := \begin{cases} \lim\limits_{\theta \to 0} \dfrac{f(t + \theta t^{1-\alpha}) - f(t)}{\theta}, & t > 0 \\ \lim\limits_{s \to 0^+} f^{(\alpha)}(s), & t = 0, \end{cases}$$

provided that the limit exists.

Properties of the Conformable Derivative

Let $\alpha \in (0, 1]$. Let f, g be α-differentiable for $t > 0$, and let $a, b \in \mathbb{R}$. Then

- $(af + bg)^{(\alpha)}(t) = af^{(\alpha)}(t) + bg^{(\alpha)}(t)$,
- $(t^b)^{(\alpha)} = bt^{b-\alpha}$,
- $(b)^{(\alpha)} = 0$,
- $(fg)^{(\alpha)}(t) = f^{(\alpha)}(t)g(t) + f(t)g^{(\alpha)}(t)$,
- $\left(\dfrac{f}{g}\right)^{(\alpha)}(t) = \dfrac{g(t)f^{(\alpha)}(t) - f(t)g^{(\alpha)}(t)}{[g(t)]^2}$, and
- if f is differentiable, then $f^{(\alpha)}(t) = t^{1-\alpha} f'(t)$.

Definition (Khalil Integral)

Let $\alpha \in (0, 1]$. The α-conformable integral of f is defined by

$$I^a_\alpha(f)(t) := \int_a^t \frac{f(\tau)}{\tau^{1-\alpha}} \mathrm{d}\tau,$$

where the integral here is the usual Riemann integral.

Delayed Model

Consider the system

$$x^{(\alpha)}(t) = Ax(t-h) + Bu(t-h)$$

where

- $x \in \mathbb{R}^n$ represents the state
- $u \in \mathbb{R}^m$ represents the control
- h is the time delay

We make the natural assumption that our system is completely observable.

Delayed Conformable Linear Quadratic Regulator (DCLQR)

System: $x^{(\alpha)}(t) = Ax(t-h) + Bu(t-h), \quad x(0) = x_0$

Cost: $J(x, u) = \dfrac{1}{2} x^T(t_f)S(t_f)x(t_f) + \dfrac{1}{2}\displaystyle\int_0^{t_f} \dfrac{[x^TQx + u^TRu](\tau)}{\tau^{1-\alpha}} \mathrm{d}\tau,$

where $S(t_f), Q \geq 0, R > 0$

Feedback Gain: $K(t) := R^{-1}B^TS(t)$

Quasi-Riccati Equation:

$$-S^{(\alpha)}(t) = A^TN^TS(t) + S(t)AN - S(t)BMR^{-1}M^TNS(t-h) + Q,$$

where $M = \dfrac{\partial u_1}{\partial u}$ and $N = \dfrac{\partial x_1}{\partial x}$

Optimal Control: $u^*(t) = -K(t)x(t)$

Optimal Cost: $J^*(x, u) = \frac{1}{2}x^T(0)S(0)x(0)$

Piecewise Solution

It can be shown that

$$M = I$$

$$N = \begin{cases} 0, & t < t_0 + h \\ I, & t \geq t_0 + h \end{cases}$$

Thus,

$$-S^{(\alpha)}(t) = \begin{cases} Q, & t < t_0 + h \\ A^TS(t) + S(t)A - S(t)BR^{-1}S(t-h) + Q, & t \geq t_0 + h \end{cases}$$

Damped Oscillator With Delay

Consider the control system

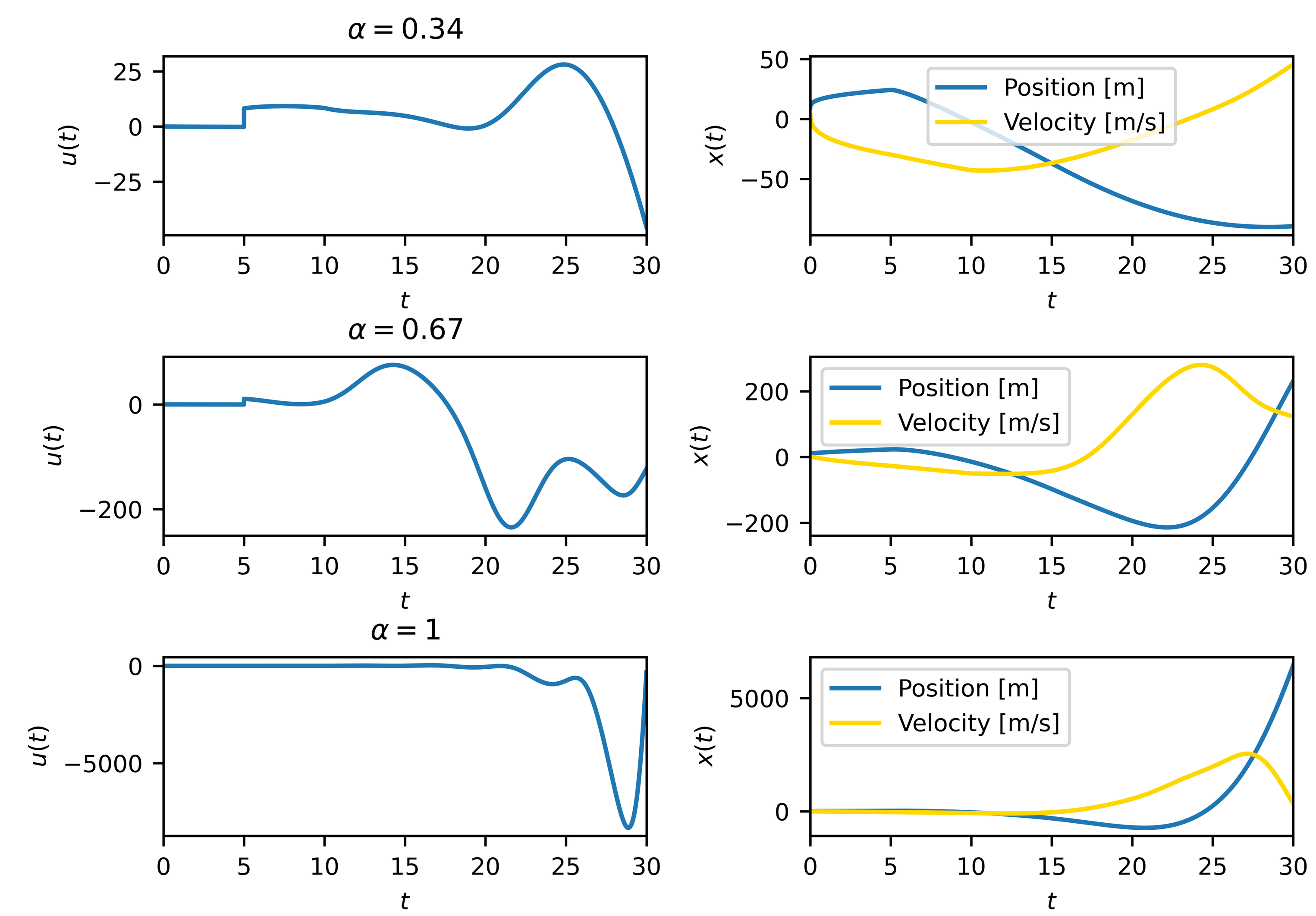
$$x^{(\alpha)}(t) = \begin{bmatrix} 0 & 1 \\ -0.64 & -0.16 \end{bmatrix} x(t-5) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t-5), \quad x_0 = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$$

associated with the quadratic cost functional

$$J(x, u) = \frac{1}{2}x^T(t_f)S(t_f)x(t_f) + \frac{1}{2}\int_0^{30} \frac{\left(x^T \begin{bmatrix} 0.001 & 0 \\ 0 & 0.001 \end{bmatrix} x + u^2\right)(\tau)}{\tau^{1-\alpha}} \mathrm{d}\tau.$$

Numerical Methods

Since $S(t_f)$ is given, we calculate $S(t),\ t_0 + h \leq t \leq t_f$ by iterating backwards in time. However, $-S^{(\alpha)}(t)$ is defined in terms of $S(t-h)$, so this method requires past information that is often unavailable. In order to sidestep this issue, we initially set $S(t) = Q$ for all $t_0 + h \leq t < t_f$. We then repeatedly solve for $S(t),\ t_0 + h \leq t \leq t_f$ backwards using the newfound S values until they sufficiently converge.



Related/Future Projects

- Conformable Kalman filter (REU group, Summer 2023)
- Conformable information filter (S. Hungerford and J. Smith, Space Grant Project 2024)
- Conformable Heart Rate Controller (with T. Cuchta and Ö. Öztürk)
- Linear quadratic pursuit-evasion games (time scales version with D. Funk and R. Williams)
- Steady-state results
- Communications models
- Conformable LQR with multiple delays in state/control
- Tracking of economic indicators

References

[1] Michael Basin and Jesus Rodriguez-Gonzalez.
A closed-form optimal control for linear systems with equal state and input delays.
Automatica, 41(5):915–920, May 2005.

[2] Pratik Adhikari, Scarlett S. Bracey, Katie A. Evans, Isidro B. Magaña, and D. Patrick O'Neal.
Lqr tracking of a delay differential equation model for the study of nanoparticle dosing strategies for cancer therapy.
In *2013 American Control Conference*, pages 2068–2073, 2013.

[3] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh.
A new definition of fractional derivative.
Journal of Computational and Applied Mathematics, 264:65–70, July 2014.

[4] Tom Cuchta, Dylan Poulsen, and Nick Wintz.
Linear quadratic tracking with continuous conformable derivatives.
European Journal of Control, 72:100808, July 2023.

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