Applications for the Conformable Information Filter

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Abstract

In this project, we offer application to our previously constructed information filter. The information filter is an algorithm used to estimate the information of a process corrupted in some way. The information filter is mathematically similar to the Kalman filter, widely used in navigation. Unlike the Kalman filter, the information filter propagates backwards in time and is more effective in smoothing. Here, our corrupted system is in terms of conformable derivative introduced by Khalil et al. in 2014. This time-weighted derivative shares many of the same properties as the classical derivative but lacks the usual semigroup property associated with the exponential. Here we offer two models for larger systems. The first model represents an aircraft in midflight tracked by radar. The second model tracks selected economic indicators over 2015-2024.

Definition (Conformable Derivative)

Let $f : [0, \infty) \to \mathbb{R}$ and let $\alpha \in (0, 1]$. Then the conformable derivative of order α of f at t is defined by

$$f^{\alpha}(t) := \begin{cases} \lim\limits_{\theta \to 0} \dfrac{f(t + \theta t^{1-\alpha}) - f(t)}{\theta}, & t > 0 \\ \lim\limits_{s \to 0^+} f^{(\alpha)}(s), & t = 0, \end{cases}$$

provided that the limit exists.

Properties of the Conformable Derivative

Let $\alpha \in (0, 1]$, f, g be $\alpha-$differentiable functions for $t > 0$, and $a, b \in \mathbb{R}$. Then

- $(af + bg)^{(\alpha)}(t) = af^{(\alpha)}(t) + bg^{(\alpha)}(t)$,
- $(t^b)^{(\alpha)} = bt^{b-\alpha}$,
- $(b)^{(\alpha)} = 0$,
- $(fg)^{(\alpha)}(t) = f^{(\alpha)}(t)g(t) + f(t)g^{(\alpha)}(t)$,
- $\left(\frac{f}{g}\right)^{(\alpha)}(t) = \frac{g(t)f^{(\alpha)}(t) - f(t)g^{(\alpha)}(t)}{[g(t)]^2}$, and
- if f is differentiable, then $f^{(\alpha)}(t) = t^{1-\alpha}f'(t)$.

Definition (Conformable Integral)

$I_{\alpha}^{a}(f)(t) = \displaystyle\int_a^t \frac{f(x)}{x^{1-\alpha}}dx$, where the integral is the usual Riemann integral and $\alpha \in (0, 1]$.

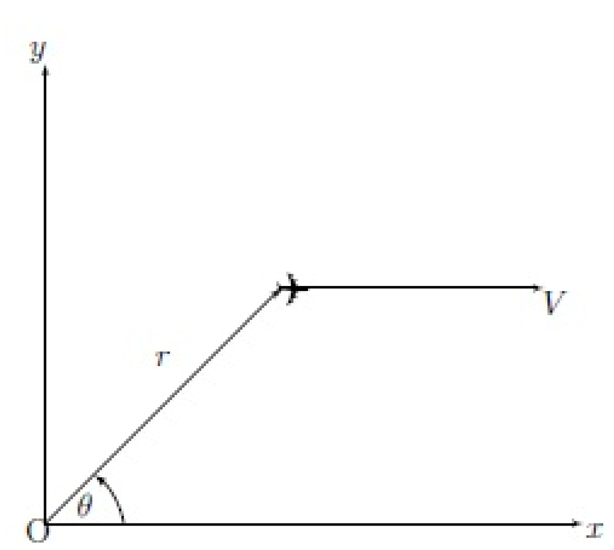
The Stochastic Model

Consider the system

$$x^{(\alpha)}(t) = A(t)x(t) + B(t)u(t) + Gw(t), \; x(t_0) = x_0$$
$$y(t) = C(t)x(t) + v(t)$$

where

- $x \in \mathbb{R}^n$ is the state,
- $u \in \mathbb{R}^m$ is the deterministic control,
- $y \in \mathbb{R}^p$ is the measurement,
- $w \in \mathbb{R}^l$ is the process noise, and
- $v \in \mathbb{R}^p$ is the measurement noise.



The Conformable Information Filter (CIF)

Backwards Error Covariance Update:
$S^{(\alpha)}(\tau) = S(\tau)A(\tau) + A^T(\tau)S(\tau) + C^T(\tau)R^{-1}C(\tau) - S(\tau)GQG^TS(\tau)$

Backwards Gain:
$K_b(\tau) = S(\tau)GQ$

Backwards Estimate Update:
$\hat{\lambda}(\tau) = S(\tau)\hat{x}(\tau)$ and $\lambda^{(\alpha)}(\tau) = (A^T(\tau) - K_b(\tau)G^T)\hat{\lambda}(\tau) - S(\tau)Bu(\tau) + C^T(\tau)(R^{-1})z(\tau)$

Smoother Gain:
$L(t) = P_f(t)S(t)(I + P_f(t)S(t))^{-1}$

Smoothed Error Covariance:
$P_s(t) = (I - L(t))P_f(t)$

Smoothed Estimate:
$\hat{x}(t) = (I - L(t))\hat{x}_f(t) + P_s(t)\hat{\lambda}(t)$

Tracking an Aircraft in Flight

Consider the model

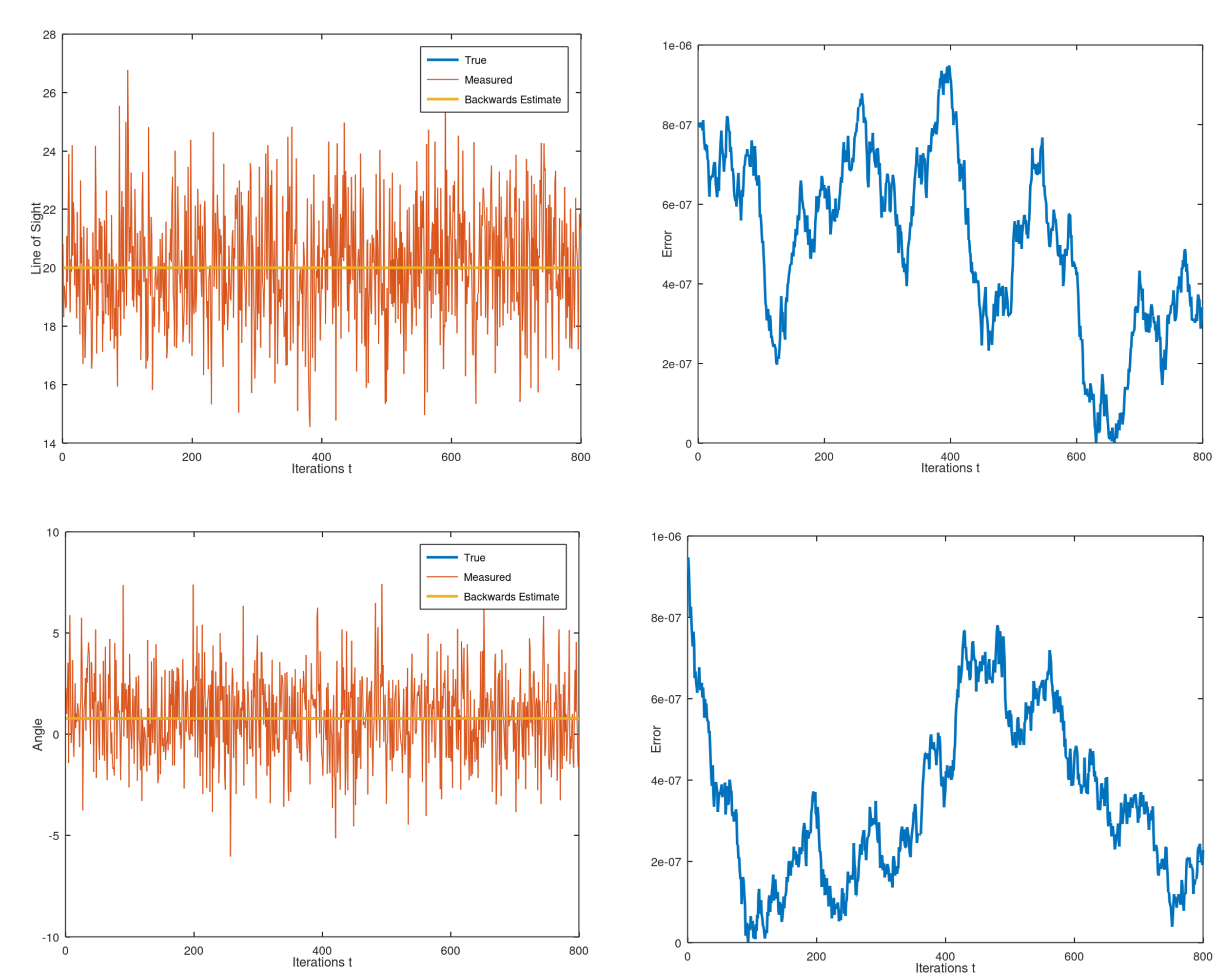
$$x^{(\alpha)}(t) = \begin{bmatrix} 1 & dt & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & dt \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} w(t), \quad x(0) = \begin{bmatrix} 20 \\ 0.01 \\ \pi/4 \\ 0.04 \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t) + v(t)$$

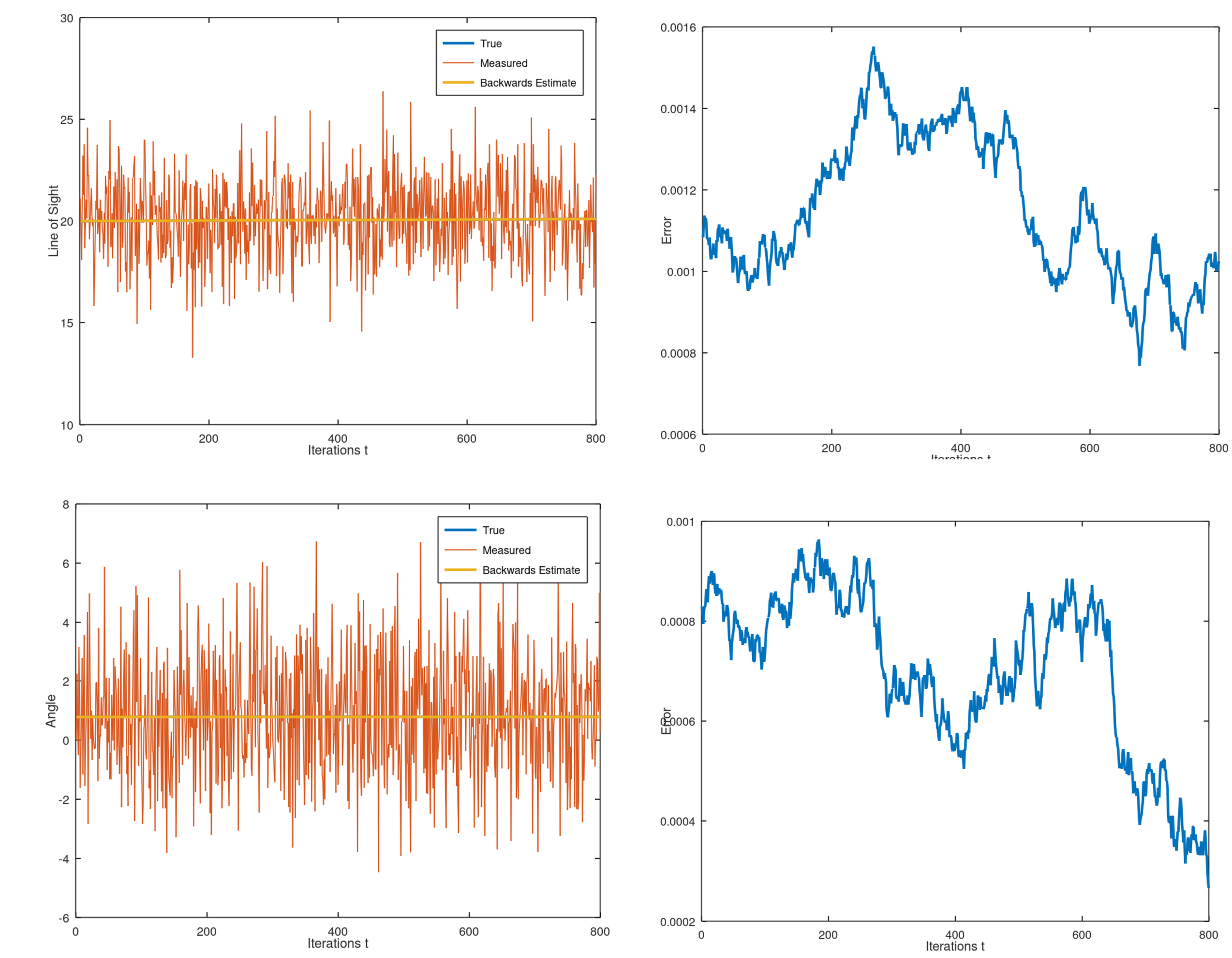
where $Q = 2^{\frac{2}{3}} \begin{bmatrix} \frac{dt^3}{3} & \frac{dt^2}{2} & 0 & 0 \\ \frac{dt^2}{2} & dt & 0 & 0 \\ 0 & 0 & \frac{dt^3}{1200} & \frac{dt^2}{800} \\ 0 & 0 & \frac{dt^2}{800} & \frac{dt}{400} \end{bmatrix}$, $R = \frac{1}{dt}\begin{bmatrix} 0.656 & 0 \\ 0 & \pi/12 \end{bmatrix}$, and

$P(0) = \frac{1}{dt}\begin{bmatrix} 0.656 & 0 & 0 & 0 \\ 0 & 0.656 & 0 & 0 \\ 0 & 0 & \pi/12 & 0 \\ 0 & 0 & 0 & \pi/12 \end{bmatrix}$. Here, we use $n = 800$ iterations.

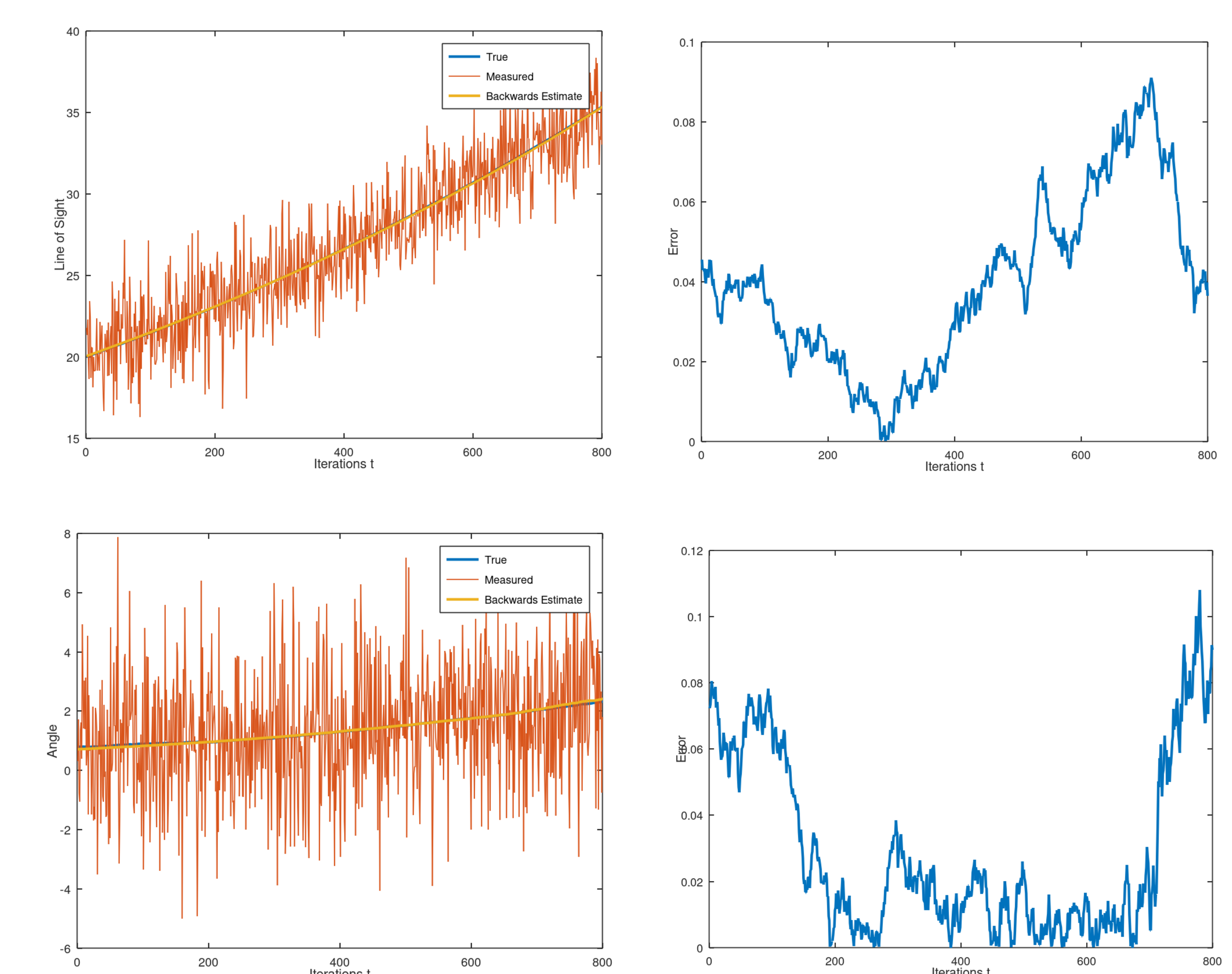
Case 1: $\alpha = 1, \mathrm{dt} = 10^{-8}$



Case 2: $\alpha = 0.67, \mathrm{dt} = 10^{-8}$



Case 3: $\alpha = 0.33, \mathrm{dt} = 10^{-11}$



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2025 Student Academic Showcase

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